Multiloop superstring amplitudes from non-minimal pure spinor formalism

Nathan Berkovits[a] **and Nikita Nekrasov**[b*]

[a] *Instituto de Física Teórica, State University of São Paulo*
Rua Pamplona 145, 01405-900, São Paulo, SP Brasil
[b] *Institut des Hautes Etudes Scientifiques*
Le Bois-Marie, 35 route de Chartres, 91440 Bures-sur-Yvette France
E-mail: `nberkovi@ift.unesp.br`, `nikita@ihes.fr`

ABSTRACT: Using the non-minimal version of the pure spinor formalism, manifestly super-Poincaré covariant superstring scattering amplitudes can be computed as in topological string theory without the need of picture-changing operators. The only subtlety comes from regularizing the functional integral over the pure spinor ghosts. In this paper, it is shown how to regularize this functional integral in a BRST-invariant manner, allowing the computation of arbitrary multiloop amplitudes. The regularization method simplifies for scattering amplitudes which contribute to ten-dimensional F-terms, i.e. terms in the ten-dimensional superspace action which do not involve integration over the maximum number of θ's.

---

[*]On leave of absence from: ITEP, Moscow, Russia

Contents

1. Introduction

Over the past six years, a new formalism combining the nice features of Ramond-Neveu-Schwarz and Green-Schwarz approaches to quantization of the superstring [1] has been developed. The formalism is called the pure spinor formalism as it involves twistor-like variables which take values in the space $\mathcal{P}$ of pure spinors in ten dimensions.

More precisely, the pure spinor formalism involves a geometric sigma model describing the maps of the worldsheet Σ to ten-dimensional super-Minkowski space, together with a somewhat unconventional *curved $\beta\gamma$-system* which describes the maps of Σ to the space $\mathcal{P}$. In the minimal pure spinor formalism, only the worldsheet fields λ^α which are holomorphic coordinates on $\mathcal{P}$ are used.

Although one can compute scattering amplitudes using the minimal pure spinor formalism, the absence of a composite b ghost in the minimal formalism makes the amplitude prescription non-conventional. In the approach of [2], picture-changing operators are used to construct a picture-raised version of the b ghost. Unfortunately, these picture-changing operators are only Lorentz-invariant up to BRST-trivial terms, so manifest Lorentz covariance is broken at intermediate stages in the computation.

A more elegant approach [3] uses the so-called Čech cohomology language, where the b ghost is viewed as a collection of Čech cochains of various degrees, from zero to three, on the space of pure spinors:

$$b = (b_\alpha) + (b_{\alpha\beta}) + (b_{\alpha\beta\gamma}) + (b_{\alpha\beta\gamma\delta}). \qquad (1.1)$$

Here α etc. label the coordinate patches $\mathcal{U}_\alpha$ on the space of pure spinors:

$$\mathcal{P} = \cup_\alpha \mathcal{U}_\alpha \qquad (1.2)$$

and one can choose the coordinate patches to be in one-to-one correspondence with the components of an unconstrained spinor, i.e. $\alpha = 1, \ldots, 16$. On the coordinate patch $\mathcal{U}_\alpha$, the component λ^α of the pure spinor is not allowed to vanish.

In this approach, manifest Lorentz invariance is broken by working with Čech cochains which are defined on the (intersections of) coordinate patches, in our case on the space $\mathcal{P}$ of pure spinors. Although the space of pure spinors is a homogeneous space of the (euclidean version of the) Lorentz group, a particular coordinate patch is not. So, it is aesthetically more appropriate to work in a formalism where the choices of coordinates on $\mathcal{P}$ are not necessary.

A well-known alternative to the Čech language in algebraic geometry is Dolbeault language where instead of the locally defined holomorphic objects, one deals with the globally defined non-holomorphic ones. In the context of two dimensional sigma models, the Dolbeault version is formulated by including the anti-holomorphic coordinates $\overline{\lambda}_\alpha$ on $\mathcal{P}$ and the fermionic coordinates $r_\alpha = d\overline{\lambda}_\alpha$. In principle, there are two options — one can treat these new coordinates as fields of the same worldsheet chirality as λ^α, as will be done in this paper following [4], or as fields of the opposite worldsheet chirality, as is more natural in the context of $(0, 2)$ models [5].

After including the non-minimal worldsheet fields $(\overline{\lambda}_\alpha, r_\alpha)$, it was shown in [4] how to construct a composite b ghost and compute superstring scattering amplitudes as in topological string theory without picture-changing insertions. The only subtlety in this non-minimal prescription is that the composite b ghost contains factors of $(\lambda^\alpha \overline{\lambda}_\alpha)^{-1}$. When functionally integrating over λ^α and $\overline{\lambda}_\alpha$ in the scattering amplitude, these $(\lambda^\alpha \overline{\lambda}_\alpha)^{-1}$ factors can cause a problem coming from the functional integration region where all components of λ^α are zero. In [4], it was shown that this problem can be avoided for amplitude computations up to two loops, but it was not shown how to resolve this problem for computations with more than two loops.

In this paper, it will be shown how to resolve this problem for arbitrary multiloop amplitudes by constructing a regularized version of the b ghost, b_ϵ, which is non-singular when $(\lambda^\alpha \overline{\lambda}_\alpha) \to 0$. The regularized b ghost will be defined as

$$b_\epsilon = e^{-\epsilon(w_\alpha \overline{w}^\alpha + \cdots)} b \qquad (1.3)$$

where ϵ is a positive constant, w_α and $\overline{w}^\alpha$ are the conjugate momenta to λ^α and $\overline{\lambda}_\alpha$, and $\ldots$ is chosen such that $(w_\alpha \overline{w}^\alpha + \cdots)$ is well-defined, i.e. gauge invariant, and BRST-trivial.

This regularization procedure can be viewed as an analogue of turning on a metric perturbation in the $(0,2)$ model. Indeed, $w_\alpha \overline{w}^\alpha$ acts essentially as the Laplacian on functions of pure spinors. If $\overline{w}^\alpha$ and w_α were fields of opposite worldsheet chirality, the term $w_\alpha \overline{w}^\alpha$ would serve as an inverse metric perturbation of a curved beta-gamma system:

$$\beta_i \overline{\partial} \gamma^i + \overline{\beta}_{\overline{i}} \partial \overline{\gamma}^{\overline{i}} + \alpha' g^{i\overline{i}} \beta_i \overline{\beta}_{\overline{i}} \tag{1.4}$$

which preserves conformal invariance for special metrics that are Ricci-flat (in the first order approximation) [6]. In our case where the fields w_α and $\overline{w}^\alpha$ have the same chirality, perturbations like (1.4) would break conformal invariance. Nevertheless, both in the context of $(0,2)$ models and in our case, perturbations like (1.4) can be made Q-exact. Thus, conformal invariance would be preserved at the level of Q-cohomology.

Also, let us mention the rôle of metric perturbations (1.4) in the context of conventional topological strings obtained by twisting $(2,2)$ supersymmetric sigma models. In particular, for the A twist one obtains the theory with the Lagrangian [7]:

$$\beta_i \overline{\partial} \gamma^i + \overline{\beta}_{\overline{i}} \partial \overline{\gamma}^{\overline{i}} + b_i \overline{\partial} c^i + \overline{b}_{\overline{i}} \partial \overline{c}^{\overline{i}} \tag{1.5}$$

that ensures that the path integral localizes onto the space of holomorphic maps of the worldsheet into the complex target space, which has an infinite radius metric in this description. This theory is well-defined (for compact targets) on genus zero, but for higher genera (in the trivial instanton sector) the zero modes of the $\beta, \overline{\beta}, b, \overline{b}$ fields need regularization. One can turn on the deformation to finite radius by adding the term:

$$\alpha' \{Q, \tfrac{1}{2} g^{i\overline{i}} (b_i \overline{\beta}_{\overline{i}} + \overline{b}_{\overline{i}} \beta_i)\} = \alpha' \left(g^{i\overline{i}} \beta_i \overline{\beta}_{\overline{i}} + \text{fermions} \right) \tag{1.6}$$

which, upon proper treatment of the coupling to worldsheet topological gravity, induces the celebrated [8] $c(\mathbf{H}_g \otimes \mathcal{T}_X))$ measure on the moduli space of Riemann surfaces of genus g.

Since (1.3) will be defined such that $b_\epsilon = b + \{Q, \Omega_\epsilon\}$ for some Ω_ϵ, BRST-invariant amplitudes are unaffected by the replacement of b with b_ϵ in the amplitude prescription of [4]. Using the regularized b ghost, the multiloop amplitude prescription in the non-minimal pure spinor formalism is therefore given by

$$A = \lim_{\epsilon \to 0} \int d^{3g-3} \tau \left\langle \prod_{r=1}^{N} \int dz_r U_r(z_r) \prod_{s=1}^{3g-3} \int (\mu_s b_\epsilon) \, \mathcal{N} \right\rangle \tag{1.7}$$

where $\int d^{3g-3} \tau \langle \prod_{r=1}^{N} \int dz_r U_r(z_r) \prod_{s=1}^{3g-3} \int (\mu_s b) \rangle$ is the usual prescription of bosonic string theory, $\mathcal{N}$ is the zero mode normalization factor defined in [4] which regularizes the functional integral over the zero modes, and the right-moving contribution to A is being ignored. Since BRST invariance implies that (1.7) is independent of the parameter ϵ, one can take the limit $\epsilon \to 0$ at the end of the computation.

The choice of the ϵ-regularization is not unique. In the present paper we give one such choice, in order to demonstrate that a completely regular expression for the amplitudes exists. Although our construction can be motivated by the considerations of (1.6), there

may well exist a much simpler regularization. At present, for generic multiloop superstring amplitude computations, it is still difficult to evaluate the limit $\epsilon \to 0$ of (1.7) since our expression for b_ϵ is rather complicated.

However, for certain special amplitudes in which not all θ^α zero modes are absorbed by the external vertex operators, the ϵ-regularization of the b ghost is unnecessary, and it is easy to take the limit $\epsilon \to 0$ of (1.7). These special amplitudes can contribute to ten-dimensional F-terms in the effective action, i.e. terms in which the superspace action involves integration over fewer than 16 θ's for N=1 D=10, or fewer than 32 θ's for N=2 D=10.

It is interesting that topological string methods are also useful for computing F-terms in the four-dimensional effective action coming from Calabi-Yau compactification [9, 8, 10]. In [4], it was shown that these lower dimensional F-term computations can be reproduced using a compactified four-dimensional version of the pure spinor formalism.

The paper is organized as follows. In section 2, the non-minimal pure spinor formalism will be reviewed. In section 3, the regularized b ghost will be constructed using the heat kernel method. In section 4, the multiloop amplitude prescription will be defined using the regularized b ghost b_ϵ, and it will be shown that this prescription simplifies for amplitudes which contribute to ten-dimensional F-terms.

2. Review of non-minimal formalism

2.1 Minimal formalism

The minimal pure spinor formalism for the superstring is constructed using the (x^m, θ^α) variables of $d = 10$ superspace where $m = 0$ to 9 and $\alpha = 1$ to 16, together with the fermionic conjugate momenta p_α. Furthermore, one introduces a bosonic spinor ghost λ^α which satisfies the pure spinor constraint

$$\lambda^\alpha \gamma^m_{\alpha\beta} \lambda^\beta = 0 \tag{2.1}$$

where $\gamma^m_{\alpha\beta}$ are the symmetric 16×16 $d = 10$ Pauli matrices. Because of the pure spinor constraint on λ^α, its conjugate momentum w_α is defined up to the gauge transformation

$$\delta w_\alpha = \Lambda^m (\gamma_m \lambda)_\alpha, \tag{2.2}$$

which implies that w_α only appears through its Lorentz current N_{mn}, ghost current J, and stress tensor T_λ. These gauge-invariant currents are defined by

$$N_{mn} = \frac{1}{2} w \gamma_{mn} \lambda, \quad J = w_\alpha \lambda^\alpha, \quad T_\lambda = w_\alpha \partial \lambda^\alpha. \tag{2.3}$$

The worldsheet action for the left-moving matter and ghost variables is

$$S = \int d^2 z \left(\frac{1}{2} \partial x^m \overline{\partial} x_m + p_\alpha \overline{\partial} \theta^\alpha - w_\alpha \overline{\partial} \lambda^\alpha \right), \tag{2.4}$$

and the right-moving variables will be ignored throughout this paper. For the Type II superstring, the right-moving variables are similar to the left-moving variables, while for the heterotic superstring, the right-moving variables are the same as in the RNS heterotic formalism.

Physical open string states in the minimal pure spinor formalism are defined as ghost-number one states in the cohomology of the nilpotent BRST operator

$$Q = \int dz \ \lambda^\alpha d_\alpha \qquad (2.5)$$

where

$$d_\alpha = p_\alpha - \frac{1}{2}\gamma^m_{\alpha\beta}\theta^\beta \partial x_m - \frac{1}{8}\gamma^m_{\alpha\beta}\gamma_{m\gamma\delta}\theta^\beta\theta^\gamma\partial\theta^\delta \qquad (2.6)$$

is the supersymmetric Green-Schwarz constraint.

Although one can compute scattering amplitudes using the minimal formalism, the lack of a composite b ghost satisfying $\{Q, b\} = T$ makes the amplitude prescription unconventional. It is easy to see that the minimal formalism does not contain such a composite b ghost since the gauge-invariant combinations of w_α in (2.3) all carry zero ghost number, so there are no gauge-invariant operators of negative ghost number.

2.2 Non-minimal worldsheet variables

As shown in [4], this difficulty can be resolved by adding non-minimal variables to the formalism which allow the construction of a composite b ghost. The new non-minimal variables consist of a bosonic pure spinor $\overline{\lambda}_\alpha$ and a constrained fermionic spinor r_α satisfying the constraints

$$\overline{\lambda}_\alpha\gamma^{\alpha\beta}_m\overline{\lambda}_\beta = 0 \quad \text{and} \quad \overline{\lambda}_\alpha\gamma^{\alpha\beta}_m r_\beta = 0. \qquad (2.7)$$

In d=10 Euclidean space where complex conjugation flips the chirality of spacetime spinors, $\overline{\lambda}_\alpha$ can be interpreted as the complex conjugate to λ^α. The worldsheet action for the non-minimal pure spinor formalism is

$$\int d^2z \ \left(\frac{1}{2}\partial x^m \overline{\partial} x_m + p_\alpha\overline{\partial}\theta^\alpha - w_\alpha\overline{\partial}\lambda^\alpha - \overline{w}^\alpha\overline{\partial}\overline{\lambda}_\alpha + s^\alpha\overline{\partial}r_\alpha \right) \qquad (2.8)$$

where $\overline{w}^\alpha$ and s^α are the conjugate momenta for $\overline{\lambda}_\alpha$ and r_α with $+1$ conformal weight.

Just as w_α can only appear in the gauge-invariant combinations

$$N_{mn} = \frac{1}{2}(w\gamma_{mn}\lambda), \quad J = w_\alpha\lambda^\alpha, \quad T_\lambda = w_\alpha\partial\lambda^\alpha, \qquad (2.9)$$

the variables $\overline{w}^\alpha$ and s^α can only appear in the combinations

$$\overline{N}_{mn} = \frac{1}{2}(\overline{w}\gamma_{mn}\overline{\lambda} - s\gamma_{mn}r), \quad \overline{J} = \overline{w}^\alpha\overline{\lambda}_\alpha - s^\alpha r_\alpha, \quad T_{\overline{\lambda}} = \overline{w}^\alpha\partial\overline{\lambda}_\alpha - s^\alpha\partial r_\alpha, \quad (2.10)$$

$$S_{mn} = \frac{1}{2}s\gamma_{mn}\overline{\lambda}, \quad S = s^\alpha\overline{\lambda}_\alpha, \qquad (2.11)$$

which are invariant under the gauge transformations

$$\delta\overline{w}^\alpha = \overline{\Lambda}^m(\gamma_m\overline{\lambda})^\alpha - \phi^m(\gamma_m r)^\alpha, \quad \delta s^\alpha = \phi^m(\gamma_m\overline{\lambda})^\alpha \qquad (2.12)$$

for arbitrary $\overline{\Lambda}^m$ and ϕ^m.

In order that the non-minimal variables do not affect the cohomology, the "minimal" pure spinor BRST operator $Q = \int dz \; \lambda^\alpha d_\alpha$ will be modified to the "non-minimal" BRST operator

$$Q_{nonmin} = \int dz \; (\lambda^\alpha d_\alpha + \overline{w}^\alpha r_\alpha) \; . \tag{2.13}$$

The new term $\int dz \overline{w}^\alpha r_\alpha$ is invariant under the gauge transformation of (2.12) and implies through the usual quartet argument that the cohomology is independent of $(\overline{\lambda}_\alpha, \overline{w}^\alpha)$ and (r_α, s^α).

The ghost-number operator in the non-minimal formalism is naturally defined as

$$\int dz(\lambda^\alpha w_\alpha - \overline{\lambda}_\alpha \overline{w}^\alpha) \tag{2.14}$$

so that λ^α carries ghost-number $+1$ and $\overline{\lambda}_\alpha$ carries ghost-number -1. The corresponding ghost-number anomaly was computed in [4] to be $+3$, so the non-minimal formalism can be treated as a critical topological string theory.

A simple way to understand the value $+3$ of the ghost number anomaly is to look at the way the measure for the non-minimal fields is defined. The issue is the zero modes. On a genus g Riemann surface, the field w_α has $11g$ zero modes, λ^α has 11 zero modes, and similarly for $\overline{w}^\alpha, \overline{\lambda}_\alpha, s^\alpha$ and r_α. The measure on w, λ zero modes is defined using the holomorphic top form Ω on the space $\mathcal{P}$ of pure spinors:

$$\mathcal{D}w_\alpha \mathcal{D}\lambda^\alpha \sim \Omega^{1-g} \, , \; \Omega = \frac{\mathrm{d}^{11}\lambda}{\lambda^3} = \lambda_+^7 \mathrm{d}\lambda_+ \wedge \mathrm{d}^{10}u_{ab} \tag{2.15}$$

where we used the local parameterization of the pure spinor in the form:

$$\lambda = \lambda_+ \left(1, u_{ab}, u_{[ab}u_{cd]}\right) .$$

The form Ω has weight $+8$ under the symmetry generated by (2.14). So, the measure factor Ω^{1-g} has charge $8(1-g)$ on the genus g surface. At the same time, the measure on $\overline{\lambda}, r$ fields is defined canonically, as the fermions r are in the antiholomorphic tangent bundle to $\mathcal{P}$. The zero modes of $\overline{\lambda}, r$ and the corresponding momenta bring a factor

$$\mathcal{D}\overline{w}^\alpha \mathcal{D}\overline{\lambda}_\alpha \mathcal{D}s^\alpha \mathcal{D}r_\alpha \sim \overline{\Omega}^{1-g} \, ,$$

where

$$\overline{\Omega} = \mathrm{d}^{11}\overline{\lambda}\mathrm{d}^{11}r$$

has charge -11 under (2.14). Thus, the total anomalous charge is $+3(g-1)$ on the genus g Riemann surface, as claimed.

2.3 Čech and Dolbeault

The addition of non-minimal variables and the construction of Q_{nonmin} can be understood as standard techniques which are used in relating Čech and Dolbeault cochains. To describe Čech cochains, first express the space of pure spinors $\mathcal{P}$ as the union of coordinate patches

U_α for $\alpha = 1$ to 16 where the component λ^α of the pure spinor is required to be non-vanishing on U_α. The analysis of anomalies of the curved $\beta\gamma$-system on the pure spinor space implies that the point $\lambda = 0$ is not in $\mathcal{P}$ [11], thus one can always find α such that a given point $\lambda \in \mathcal{P}$ belongs to the coordinate patch $\mathcal{U}_\alpha$. So $\mathcal{P} = \cup_\alpha \mathcal{U}_\alpha$.

The Čech k-cochain is an object $\psi_{\alpha_0\alpha_1\ldots\alpha_k}$ which is holomorphic on the intersection

$$\mathcal{U}_{\alpha_0\alpha_1\ldots\alpha_k} = \mathcal{U}_{\alpha_0} \cap \mathcal{U}_{\alpha_1} \cap \ldots \cap \mathcal{U}_{\alpha_k} \tag{2.16}$$

and which obeys

$$(\delta\psi)_{\alpha_0\alpha_1\ldots\alpha_k} \equiv \psi_{\alpha_1\alpha_2\ldots\alpha_k} - \psi_{\alpha_0\alpha_2\ldots\alpha_k} + \cdots (-1)^k \psi_{\alpha_0\alpha_1\ldots\alpha_{k-1}} = 0 . \tag{2.17}$$

The standard way to relate Čech and Dolbeault cochains is to use the so-called *partition of unity* (cf. [5]). In the case of the space of pure spinors it can be taken to be:

$$\rho_\alpha = \frac{1}{(\overline{\lambda}\lambda)}\overline{\lambda}_\alpha \lambda^\alpha \tag{2.18}$$

where the functions ρ_α vanish outside the corresponding domains $\mathcal{U}_\alpha$, and they sum to unity

$$\sum_{\alpha=1}^{16} \rho_\alpha = 1.$$

Note that $(\overline{\lambda}\lambda)$ denotes $\sum_{\beta=1}^{16} \overline{\lambda}_\beta \lambda^\beta$ and repeated indices are not assumed to be summed over in this subsection.

Now, given a Čech cochain satisfying (2.17), one can define the corresponding Dolbeault cocycle, i.e. a $\overline{\partial}$-closed differential form of type $(0, p-1)$:

$$\widehat{\psi} = \frac{1}{p!} \sum_{\alpha_1,\ldots,\alpha_p} \psi_{\alpha_1\ldots\alpha_p} \rho_{\alpha_1} \overline{\partial}\rho_{\alpha_2} \wedge \ldots \wedge \overline{\partial}\rho_{\alpha_p}. \tag{2.19}$$

An important generalization of this well-known construction consists of replacing the function valued cochains $(\psi_{\ldots})$ by cochains which take values in some (super)commutative algebra, of even complex, and by replacing the operator $\overline{\partial}$ by the general operator $\overline{\partial} + Q$ where Q is the differential in this complex. The resulting globally defined form $\widehat{\psi}$ obeys

$$(\overline{\partial} + Q)\widehat{\psi} = 0$$

iff the Čech cochain verifies

$$(\delta + Q)\psi = 0.$$

To compare with the non-minimal formalism described in the previous subsection, define

$$Q = \sum_{\alpha=1}^{16} \int dz \; \lambda^\alpha d_\alpha \quad \text{and} \quad \overline{\partial} = \sum_{\alpha=1}^{16} \int dz \; \overline{w}^\alpha r_\alpha \tag{2.20}$$

so that $Q_{nonmin}\widehat{\psi} = (\overline{\partial} + Q)\widehat{\psi}$. Using this definition, one finds in (2.19) that

$$\overline{\partial}\rho_\alpha = \frac{(\overline{\lambda}\lambda)r_\alpha - (r\lambda)\overline{\lambda}_\alpha}{(\lambda\overline{\lambda})^2}\lambda^\alpha. \tag{2.21}$$

2.4 Construction of b ghost

Although there is no globally defined operator in the minimal formalism satisfying $\{Q, b\} = T$, a b ghost can be constructed in the non-minimal formalism using the operators $[G^\alpha, H^{[\alpha\beta]}, K^{[\alpha\beta\gamma]}, L^{[\alpha\beta\gamma\delta]}]$ which carry zero ghost-number and satisfy [4, 2]

$$\{Q, G^\alpha\} = \lambda^\alpha T, \quad [Q, H^{[\alpha\beta]}] = \lambda^{[\alpha} G^{\beta]}, \quad \{Q, K^{[\alpha\beta\gamma]}\} = \lambda^{[\alpha} H^{\beta\gamma]},$$
$$[Q, L^{[\alpha\beta\gamma\delta]}] = \lambda^{[\alpha} K^{\beta\gamma\delta]}, \quad \lambda^{[\alpha} L^{\beta\gamma\delta\kappa]} = 0. \tag{2.22}$$

As discussed in [3], this construction can be naturally understood in Čech language by defining

$$b = (b_\alpha) + (b_{\alpha\beta}) + (b_{\alpha\beta\gamma}) + (b_{\alpha\beta\gamma\delta}) \tag{2.23}$$

where $[(b_\alpha), (b_{\alpha\beta}), (b_{\alpha\beta\gamma}), (b_{\alpha\beta\gamma\delta})]$ are Čech cochains of degree zero to three defined by

$$(b_\alpha) = \frac{G^\alpha}{\lambda^\alpha}, \quad (b_{\alpha\beta}) = \frac{H^{[\alpha\beta]}}{\lambda^\alpha \lambda^\beta}, \quad (b_{\alpha\beta\gamma}) = \frac{K^{[\alpha\beta\gamma]}}{\lambda^\alpha \lambda^\beta \lambda^\gamma}, \quad (b_{\alpha\beta\gamma\delta}) = \frac{L^{[\alpha\beta\gamma\delta]}}{\lambda^\alpha \lambda^\beta \lambda^\gamma \lambda^\delta}. \tag{2.24}$$

It is not difficult to show that (2.22) implies that $\{Q + \delta, b\} = T$. Using the methods of the previous subsection, the corresponding globally defined Dolbeault form is therefore

$$b = \frac{\overline{\lambda}_\alpha G^\alpha}{(\overline{\lambda}\lambda)} + \frac{\overline{\lambda}_\alpha r_\beta H^{[\alpha\beta]}}{(\overline{\lambda}\lambda)^2} - \frac{\overline{\lambda}_\alpha r_\beta r_\gamma K^{[\alpha\beta\gamma]}}{(\overline{\lambda}\lambda)^3} - \frac{\overline{\lambda}_\alpha r_\beta r_\gamma r_\delta L^{[\alpha\beta\gamma\delta]}}{(\overline{\lambda}\lambda)^4}, \tag{2.25}$$

which satisfies $\{Q_{nonmin}, b\} = T$.

Finally, to construct a b_{nonmin} ghost satisfying $\{Q_{nonmin}, b_{nonmin}\} = T_{nonmin}$ where $T_{nonmin} = T + \overline{w}^\alpha \partial \overline{\lambda}_\alpha - s^\alpha \partial r_\alpha$, one defines $b_{nonmin} = b + s^\alpha \partial \overline{\lambda}_\alpha$. Plugging in the explicit form of the operators $[G^\alpha, H^{[\alpha\beta]}, K^{[\alpha\beta\gamma]}, L^{[\alpha\beta\gamma\delta]}]$, one finds that [4]

$$b_{nonmin} = s^\alpha \partial \overline{\lambda}_\alpha + \frac{\overline{\lambda}_\alpha (2\Pi^m (\gamma_m d)^\alpha - N_{mn}(\gamma^{mn}\partial\theta)^\alpha - J_\lambda \partial\theta^\alpha - \frac{1}{4}\partial^2\theta^\alpha)}{4(\overline{\lambda}\lambda)}$$
$$+ \frac{(\overline{\lambda}\gamma^{mnp}r)(d\gamma_{mnp}d + 24N_{mn}\Pi_p)}{192(\overline{\lambda}\lambda)^2} - \frac{(r\gamma_{mnp}r)(\overline{\lambda}\gamma^m d)N^{np}}{16(\overline{\lambda}\lambda)^3}$$
$$+ \frac{(r\gamma_{mnp}r)(\overline{\lambda}\gamma^{pqr}r)N^{mn}N_{qr}}{128(\overline{\lambda}\lambda)^4}. \tag{2.26}$$

Note that throughout the rest of this paper, the subscript *nonmin* will be dropped from b_{nonmin} and Q_{nonmin}. Instead, we shall sometimes use the subscript *min*, in order to stress the use of the minimal formalism.

2.5 Amplitude prescription

Using the composite b ghost defined in (2.26), the naive topological prescription for N-point g-loop amplitudes is

$$A = \int_{\mathcal{M}_{g,N}} d^{3g-3}\tau \left\langle \prod_{j=1}^{3g-3} \left(\int dw_j \mu_j(w_j) b(w_j) \right) \prod_{r=1}^N dz_r U(z_r) \right\rangle \tag{2.27}$$

– 8 –

where τ_j are the complex Teichmuller parameters, μ_j are the associated Beltrami differentials, $\int dz U(z)$ are the BRST-invariant integrated vertex operators which can be assumed to be independent of the non-minimal fields, $\langle \ \rangle$ denotes functional integration over the worldsheet fields, and the right-moving contribution to A is being ignored. Since the ghost-number anomaly of the non-minimal formalism is $+3$, this topological prescription is reasonable. However, as explained in [4], there are two subtleties with this amplitude prescription which are associated with the functional integration over the pure spinors.

The first subtlety is that the bosonic ghosts $(\lambda^\alpha, \overline{\lambda}_\alpha)$ have 22 non-compact zero modes, and integration over these zero modes produces infinities when $\lambda \to \infty$. Similarly, the conjugate momenta $(w_\alpha, \overline{w}^\alpha)$ have $22g$ non-compact zero modes on a genus g surface which also produce infinities when $w \to \infty$. Fortunately, these infinities are cancelled by zeros coming from integration over the zero modes of the fermionic variables $(\theta^\alpha, r_\alpha)$ and their conjugate momenta (p_α, s^α).

The 0/0 factors coming from integration over the bosonic and fermionic zero modes can be regularized by inserting an operator $\mathcal{N} = e^{\{Q, \chi\}}$ into the integral over the zero modes. Since $\mathcal{N} = 1 + \{Q, \Omega\}$ for some Ω, the choice of χ does not affect BRST-invariant expressions. A convenient choice for χ is [4]

$$\chi = -\overline{\lambda}_\alpha \theta^\alpha - \sum_{I=1}^{g} (\frac{1}{2} N_{mn}^I S^{mnI} + J^I S^I) \tag{2.28}$$

where $[N_{mn}^I, J^I, S_{mn}^I, S^I]$ are the zero modes of $[N_{mn}, J, S_{mn}, S]$ of (2.9) and (2.10) obtained by integrating these currents around the Ith a-cycle, e.g. $N_{mn}^I = \oint_{a_I} dz N_{mn}(z)$.

With this choice,

$$\mathcal{N} = \exp \left(-\overline{\lambda}_\alpha \lambda^\alpha - r_\alpha \theta^\alpha - \sum_{I=1}^{g} \left[\frac{1}{2} N_{mn}^I \overline{N}^{mnI} + J^I \overline{J}^I + \frac{1}{4} S_{mn}^I \ d^I \gamma^{mn} \lambda + S^I \ \lambda^\alpha d_\alpha^I \right] \right), \tag{2.29}$$

which imposes an exponential cutoff for the non-compact bosonic zero modes. Although $\mathcal{N}$ is not manifestly invariant under spacetime supersymmetry transformations or under modular transformations of the genus g worldsheet, it is easy to show that it changes by BRST-trivial quantities under these transformations. Since $\mathcal{N}$ only involves worldsheet zero modes, these BRST-trivial quantities are harmless and cannot produce surface terms in the integral over the Teichmuller moduli. Note that the regulator $\mathcal{N}$ is somewhat similar to the *projection form* used in topological gauge theory to fix the fermionic gauge invariance (see, e.g. [12]).

The second subtlety with (2.27) is more difficult to resolve and comes from the singularities in the b ghost of (2.26) when $(\overline{\lambda}\lambda) \to 0$. Since the measure factor for the pure spinors converges like $(\overline{\lambda}\lambda)^{11}$ when $(\overline{\lambda}\lambda) \to 0$, these singularities are dangerous if they combine to diverge as fast as $(\overline{\lambda}\lambda)^{-11}$. Since each b ghost can diverge like $(\overline{\lambda}\lambda)^{-3}$, there are potential problems with the amplitude prescription when there are more than three b ghosts, i.e. when $g_{\dot{\iota}}2$.

As explained in [4], this second subtlety is related to the existence of the operator

$$\xi = \frac{\overline{\lambda}_\alpha \theta^\alpha}{\lambda^\beta \overline{\lambda}_\beta + r_\beta \theta^\beta} = (\overline{\lambda}\theta) \sum_{n=1}^{11} \frac{(-r\theta)^{n-1}}{(\lambda\overline{\lambda})^n} \qquad (2.30)$$

which satisfies $\{Q, \xi\} = 1$ and diverges like $(\lambda\overline{\lambda})^{-11}$. Since $QV = 0$ implies that $Q(\xi V) = V$, the existence of the operator ξ naively implies that the BRST cohomology is trivial. So if operators which diverge as $(\lambda\overline{\lambda})^{-11}$ are allowed in the Hilbert space, the BRST cohomology becomes trivial and one should expect to encounter problems in correlation functions and scattering amplitudes.

It is instructive to give the Čech picture of the ξ-operator. It is given by the inhomogeneous cochain:

$$\xi = \left(\frac{\theta^\alpha}{\lambda^\alpha}\right) + \left(\frac{\theta^\alpha \theta^\beta}{\lambda^\alpha \lambda^\beta}\right) + \ldots + \left(\frac{\theta^1 \ldots \theta^{16}}{\lambda^1 \ldots \lambda^{16}}\right) \qquad (2.31)$$

which obeys:

$$\delta\xi + Q_{min}\xi = 1 . \qquad (2.32)$$

In the following section, this second subtlety will be resolved by constructing a regularized version of the b ghost which is non-singular when $(\overline{\lambda}\lambda) \to 0$. After replacing the b ghost with its regularized version, it will be possible to use the prescription of (2.27) to compute arbitrary multiloop amplitudes.

3. Regularization of b Ghost

3.1 Regularization of local operators

In our regularization method, we will deal with operators such as the b ghost which involve singular-looking expressions like

$$\frac{1}{(\overline{\lambda}\lambda)^l}$$

with $l < 11$. It is important to show that correlation functions of such operators are finite in the pure spinor $\beta\gamma$ system. To this end we shall produce now a Q-invariant regularization, which does not change the Q-cohomology class of an operator, while making it explicitly non-singular.

The idea of this regularization can be first explained in the example of quantum mechanics, where we do not deal with the issue of Q-invariance. So let us first study the quantum mechanics of a particle with zero Hamiltonian in a phase space with the coordinates (p_m, q^m) for $m = 1$ to d. Suppose we face the following problem:

Let $\mathcal{O}_l(q)$ be a function which has a pole of order l at the point $q = 0$. Then, naively, the correlation function

$$\langle \mathcal{O}_{l_1} \mathcal{O}_{l_2} \ldots \mathcal{O}_{l_p} \rangle = \int \mathrm{d}^d q \, \mathcal{O}_{l_1}(q) \mathcal{O}_{l_2}(q) \ldots \mathcal{O}_{l_p}(q) \qquad (3.1)$$

is singular when $l_1 + l_2 + \cdots + l_p \geq d$.

Now imagine adding the Hamiltonian $\epsilon^2 \Delta = \epsilon^2 g^{mn} p_m p_n$ where ϵ is a constant. As long as the operators $\mathcal{O}_{l_k}(q)$ are separated and satisfy the individual conditions $l_k < d$, the smearing due to the heat kernel evolution will make them non-singular. Indeed, we have the heat kernel regularization of local operators:

$$
\begin{aligned}
\mathcal{O}_l(q) \mapsto \mathcal{O}_{l,\epsilon}(q) &= e^{\epsilon^2 \Delta} \mathcal{O}_l(q) = e^{-\epsilon^2 g^{mn} p_m p_n} \mathcal{O}_l(q) \\
&= \frac{1}{(4\pi)^{\frac{d}{2}}} \int d^d f \ e^{-f^2} e^{i\epsilon f^m p_m} \mathcal{O}_l(q) = \frac{1}{(4\pi)^{\frac{d}{2}}} \int d^d f \ e^{-f^2} \mathcal{O}_l(q + \epsilon f) \\
&= \frac{1}{(4\pi\epsilon^2)^{\frac{d}{2}}} \int d^d q' \ e^{-\frac{1}{\epsilon}(q-q')^2} \mathcal{O}_l(q')
\end{aligned}
\tag{3.2}
$$

where $q' = q + \epsilon f$. Note that in the above derivation, g_{mn} is assumed to be constant and the momenta p_m are treated as operators.

Now, as long as $l < d$, the integral in (3.2) converges, and is non-singular at $q = 0$:

$$
\mathcal{O}_{l,\epsilon}(q \to 0) \propto \epsilon^{-l} < \infty.
\tag{3.3}
$$

So the heat kernel regularization has "smeared out" the singularity of $\mathcal{O}_l(q)$ at $q = 0$. A similar regularization will be now proposed for observables on the pure spinor space such that

$$
\mathcal{O}(\lambda, \overline{\lambda}) \mapsto \mathcal{O}_\epsilon(\lambda, \overline{\lambda}) = e^{\epsilon^2 \Delta} \mathcal{O}(\lambda, \overline{\lambda}).
\tag{3.4}
$$

3.2 Regularization in pure spinor space

Since w_α and $\overline{w}^\alpha$ are the conjugate momenta to λ^α and $\overline{\lambda}_\alpha$, a naive guess for the Laplacian on pure spinor space is $\Delta = w_\alpha \overline{w}^\alpha$. So the naive generalization of (3.2) to pure spinors is

$$
\begin{aligned}
\mathcal{O}_\epsilon(\lambda, \overline{\lambda}) &= \frac{1}{(4\pi)^{11}} \int d^{11} f d^{11} \overline{f} \ e^{-f^\alpha \overline{f}_\alpha} e^{i\epsilon(f^\alpha w_\alpha + \overline{f}_\alpha \overline{w}_\alpha)} \mathcal{O}(\lambda, \overline{\lambda}) \\
&= \frac{1}{(4\pi)^{11}} \int d^{11} f d^{11} \overline{f} \ e^{-f^\alpha \overline{f}_\alpha} \mathcal{O}(\lambda', \overline{\lambda}')
\end{aligned}
\tag{3.5}
$$

where $\lambda' = \lambda + \epsilon f$, $\overline{\lambda}' = \overline{\lambda} + \epsilon \overline{f}$, and f^α and $\overline{f}_\alpha$ are pure spinors. However, since $\lambda' = \lambda + \epsilon f$ is not necessarily a pure spinor, this definition needs to be modified. The problem is that w_α and $\overline{w}^\alpha$ are not gauge-invariant under (2.2) and (2.12), so their commutation relations with λ^α and $\overline{\lambda}_\alpha$ are not well-defined.

Gauge-invariant versions of w_α and $\overline{w}^\alpha$ can be defined as

$$
\begin{aligned}
W_\alpha &= (\lambda \overline{f})^{-1} \left(-\frac{1}{8} (\lambda \gamma^{mn} w)(\gamma_{mn} \overline{f})_\alpha - \frac{1}{4} (\lambda w) \overline{f}_\alpha \right), \\
\overline{W}^\alpha &= (f \overline{\lambda})^{-1} \left(-\frac{1}{8} (\overline{\lambda} \gamma^{mn} \overline{w})(\gamma_{mn} f)^\alpha - \frac{1}{4} (\overline{\lambda} \overline{w}) f^\alpha \right)
\end{aligned}
\tag{3.6}
$$

where f^α and $\overline{f}_\alpha$ are constant pure spinors. Using the identity

$$
\delta_\beta^\gamma \delta_\alpha^\delta = \frac{1}{2} \gamma_{\alpha\beta}^m \gamma_m^{\gamma\delta} - \frac{1}{8} (\gamma^{mn})_\alpha{}^\gamma (\gamma_{mn})_\beta{}^\delta - \frac{1}{4} \delta_\alpha^\gamma \delta_\beta^\delta
\tag{3.7}
$$

– 11 –

which can be proven by contracting both sides of (3.7) with $\gamma_p^{\alpha\beta}$, $\gamma_{pqr}^{\alpha\beta}$ or $\gamma_{pqrst}^{\alpha\beta}$, one finds that

$$(\lambda\overline{f})w_\alpha = \frac{1}{2}(w\gamma^m\overline{f})(\gamma_m\lambda)_\alpha - \frac{1}{8}(\lambda\gamma^{mn}w)(\gamma_{mn}\overline{f})_\alpha - \frac{1}{4}(w\lambda)\overline{f}_\alpha. \tag{3.8}$$

So in the gauge $w\gamma^m\overline{f} = \overline{w}\gamma^m f = 0$, $W_\alpha = w_\alpha$ and $\overline{W}^\alpha = \overline{w}^\alpha$.

Note that W_α can be written more compactly as:

$$W_\alpha = \frac{1}{4}(N\!\!\!/\lambda_\alpha^{-1} - J\lambda_\alpha^{-1}) \tag{3.9}$$

where

$$\lambda^{-1} = \frac{1}{\lambda\overline{f}}\overline{f} \tag{3.10}$$

Although the global gauge-invariant differential operators on $\mathcal{P}$ are polynomials in N_{mn} and J, which act by "rotations" which preserve the point $\lambda = 0$, the parameters of the "rotations" in (3.9), (3.10) are singular at $\lambda = 0$, allowing the operators W_α to shift the "bad" point $\lambda = 0$ in what follows.

One can therefore define

$$\mathcal{O}_\epsilon(\lambda,\overline{\lambda}) = \frac{1}{(4\pi)^{11}} \int d^{11}f d^{11}\overline{f}\ e^{-f^\alpha\overline{f}_\alpha} e^{i\epsilon(f^\alpha W_\alpha + \overline{f}_\alpha \overline{W}^\alpha)} \mathcal{O}(\lambda,\overline{\lambda}) \tag{3.11}$$

as a gauge-invariant version of (3.5). Although W_α of (3.6) needs to be normal-ordered, the normal-ordering ambiguity commutes with λ^α and therefore does not affect the definition of (3.11). Using the OPE's of N^{mn} and J with λ^α, $\mathcal{O}_\epsilon(\lambda\overline{\lambda})$ can be expressed as:

$$\mathcal{O}_\epsilon(\lambda,\overline{\lambda}) = \frac{1}{(4\pi)^{11}} \int d^{11}f d^{11}\overline{f}\ e^{-f^\alpha\overline{f}_\alpha} \mathcal{O}(\lambda',\overline{\lambda}') \tag{3.12}$$

where

$$\lambda'^\alpha = e^{i\epsilon f^\beta W_\beta}\lambda^\alpha = \lambda^\alpha + \epsilon f^\alpha - \frac{[(\lambda+\epsilon f)\gamma^m(\lambda+\epsilon f)](\gamma_m\overline{f})^\alpha}{4(\lambda+\epsilon f)^\beta\overline{f}_\beta},$$

$$\overline{\lambda}'_\alpha = e^{i\epsilon\overline{f}_\beta\overline{W}^\beta}\overline{\lambda}^\alpha = \overline{\lambda}_\alpha + \epsilon\overline{f}_\alpha - \frac{[(\overline{\lambda}+\epsilon\overline{f})\gamma^m(\overline{\lambda}+\epsilon\overline{f})](\gamma_m f)_\alpha}{4(\overline{\lambda}+\epsilon\overline{f})_\beta f^\beta}. \tag{3.13}$$

Note that $\int d^{11}f d^{11}\overline{f}$ denotes $\int \Omega\overline{\Omega}(f^\alpha\overline{f}_\alpha)^3$ where $\Omega = \frac{d^{11}f}{f^3}$ and $\overline{\Omega} = \frac{d^{11}\overline{f}}{\overline{f}^3}$ are the holomorphic and antiholomorphic top-forms on the space of pure spinors. As will be seen in the following subsection, the additional factor of $(f^\alpha\overline{f}_\alpha)^3$ in the integration measure is absent in the BRST-invariant generalization of (3.12).

It is easy to check that λ' and $\overline{\lambda}'$ of (3.13) are pure spinors satisfying $\lambda'\gamma_m\lambda' = \overline{\lambda}'\gamma_m\overline{\lambda}' = 0$. In fact, one way to derive (3.13) is to require that λ' is a pure spinor and that $\lambda'^\alpha = \lambda^\alpha + \epsilon f^\alpha + \Omega^m(\gamma_m\overline{f})^\alpha$ for some Ω^m. The additional term proportional to Ω^m comes from the commutation relation

$$[W_\alpha, \lambda^\beta] = \delta_\alpha^\beta - \frac{1}{2}(\gamma^m\lambda)_\alpha(\gamma_m\lambda^{-1})^\beta. \tag{3.14}$$

Another way to understand (3.13) is to use the parameterization

$$\lambda^\alpha = \lambda^+ \left(1, u_{ab}, u_{[ab}u_{cd]}\right) \tag{3.15}$$

of a pure spinor. Then, given two pure spinors, λ and $\epsilon f = \epsilon f^+\left(1, \phi_{ab}, \phi_{[ab}\phi_{cd]}\right)$, one can construct the third one by taking

$$\lambda'^\alpha = \left(\lambda^+ + \epsilon f^+\right) \cdot \left(1, u'_{ab}, u'_{[ab}u'_{cd]}\right) \tag{3.16}$$

where

$$u'_{ab} = \frac{\lambda^+ u_{ab} + \epsilon f^+ \phi_{ab}}{\lambda^+ + \epsilon f^+}. \tag{3.17}$$

This "addition of pure spinors" is equivalent to (3.13) for $\overline{f} = (1, 0, 0)$.

It will now be argued that $\mathcal{O}_\epsilon(\lambda, \overline{\lambda})$ in (3.12) is well-defined for all values of λ if one assumes that $\mathcal{O}(\lambda', \overline{\lambda}') \sim (\lambda'\overline{\lambda}')^{-n}$ where $0 \le n < 11$. Firstly, note that when $\lambda^\alpha \to 0$ and $\overline{\lambda}_\alpha \to 0$, (3.13) implies that $\lambda'^\alpha \to f^\alpha$ and $\overline{\lambda}'_\alpha \to \overline{f}_\alpha$. So as in the quantum-mechanical example, the regularization $\mathcal{O}(\lambda, \overline{\lambda}) \mapsto \mathcal{O}_\epsilon(\lambda, \overline{\lambda})$ smears out the singularity of $\mathcal{O}(\lambda, \overline{\lambda})$ at $\lambda = \overline{\lambda} = 0$. Since $\mathcal{O}(\lambda, \overline{\lambda})$ diverges slower than $(\lambda\overline{\lambda})^{-11}$, there are no singularites in $\mathcal{O}_\epsilon(\lambda, \overline{\lambda})$ when $\lambda = \overline{\lambda} = 0$.

Secondly, note that when $(\lambda + \epsilon f)^\beta \overline{f}_\beta \to 0$, (3.13) implies that λ' diverges. However, since $\mathcal{O}(\lambda', \overline{\lambda}') \sim (\lambda'\overline{\lambda}')^{-n}$ for $n \ge 0$, $\mathcal{O}_\epsilon(\lambda, \overline{\lambda})$ remains finite when λ' diverges.

Finally, suppose that λ is chosen such that $\lambda'^\alpha = e^{i\epsilon f^\beta W_\beta}\lambda^\alpha$ vanishes. For (3.12) to be well-defined, it is necessary that the measure factor $d^{11}f d^{11}\overline{f}$ converges faster than $(\lambda'\overline{\lambda}')^n$ when $\lambda' \to 0$. It will be useful to consider separately the cases when $\lambda^\alpha = 0$ and when λ^α is non-zero. When $\lambda^\alpha = 0$, (3.13) implies that $\lambda' = \epsilon f$. So $d^{11}f d^{11}\overline{f} = \epsilon^{22} d^{11}\lambda' d^{11}\overline{\lambda}'$, which converges as $(\lambda'\overline{\lambda}')^{11}$ near $\lambda' = 0$. When λ^α is non-zero, one can choose a Lorentz frame in which λ^+ is non-zero and $u_{ab} = 0$ in (3.15). Using the parameterization of (3.16), $\lambda'^\alpha \to 0$ implies that $(\lambda^+ + \epsilon f^+) \to 0$ with u'_{ab} held fixed. Since (3.17) implies that $\phi_{ab} \to (\lambda^+ + \epsilon f^+)\frac{u'_{ab}}{\epsilon f^+}$, one finds that $d^{11}f d^{11}\overline{f}$ converges like $|\lambda^+ + \epsilon f^+|^{22}$ when $\lambda'^\alpha \to 0$, which is fast enough to cancel the $(\lambda'\overline{\lambda}')^{-n}$ divergence if $n < 11$.

3.3 BRST-invariant regulator

To make the regularization method of (3.11) BRST-invariant, it is convenient to introduce constant bosonic pure spinors f^α and $\overline{f}_\alpha$, and constant constrained fermions g^α and $\overline{g}_\alpha$, satisfying

$$g\gamma^m f = 0, \quad \overline{g}\gamma^m\overline{f} = 0, \quad f\gamma^m f = 0, \quad \overline{f}\gamma^m\overline{f} = 0, \tag{3.18}$$

and to define $[f^\alpha, \overline{f}_\alpha, g^\alpha, \overline{g}_\alpha]$ to transform under BRST transformations as

$$[Q, f^\alpha] = 0, \quad [Q, \overline{f}_\alpha] = \overline{g}_\alpha, \quad \{Q, g^\alpha\} = f^\alpha, \quad \{Q, \overline{g}_\alpha\} = 0. \tag{3.19}$$

The constraints of (3.18) imply that f^α, $\overline{f}_\alpha$, g^α and $\overline{g}_\alpha$ each have eleven independent components.

Geometrically, the Q-operator (3.19) can be identified with the operator

$$\overline{\partial} + \iota_E$$

acting on the space $\Omega^{\bullet,\bullet}(\mathcal{P})$ of all differential forms on the space of pure spinors. Here $E = \lambda\frac{\partial}{\partial\lambda}$ is the holomorphic Euler vector field. The familiar $U(1)$ action on $\mathcal{P}$ is generated by the vector field $U = -i(E - \overline{E})$. In this picture $g^\alpha = df^\alpha$, $\overline{g}_\alpha = d\overline{f}_\alpha$. The operator (3.19) can be viewed as a "half" of the $U(1)$ equivariant differential $d + \iota_U \sim Q + \overline{Q}$.

We have

$$Q(\overline{f}_\alpha g^\alpha) = \overline{f}_\alpha f^\alpha + \overline{g}_\alpha g^\alpha$$

which is the $U(1)$-equivariant symplectic form on $\mathcal{P}$. Furthermore, if one defines

$$W_\alpha = (\lambda\overline{f})^{-1}\left(\frac{1}{4}N_{mn}(\gamma^{mn}\overline{f})_\alpha - \frac{1}{4}J\overline{f}_\alpha\right),$$

$$V^\alpha = (f\overline{\lambda})^{-1}(\frac{1}{4}S_{mn}(\gamma^{mn}f)^\alpha - \frac{1}{4}Sf^\alpha), \tag{3.20}$$

one finds that

$$[Q,W_\alpha] = (\lambda\overline{f})^{-1}\left(\frac{1}{8}(\lambda\gamma_{mn}d)(\gamma^{mn}\overline{f})_\alpha + \frac{1}{4}(\lambda d)\overline{f}_\alpha + \frac{1}{4}N_{mn}(\gamma^{mn}\overline{g})_\alpha - \frac{1}{4}J\overline{g}_\alpha\right) - \frac{(\lambda\overline{g})}{(\lambda\overline{f})}W_\alpha,$$

$$\{Q,V^\alpha\} = (f\overline{\lambda})^{-1}\left(\frac{1}{4}\overline{N}^{mn}(\gamma^{mn}f)^\alpha - \frac{1}{4}\overline{J}f^\alpha\right) - \frac{(fr)}{(f\overline{\lambda})}V^\alpha. \tag{3.21}$$

Up to terms involving fermions, it is easy to verify that $f^\alpha W_\alpha + \overline{f}_\alpha\overline{W}_\alpha = Q(g^\alpha W_\alpha + \overline{f}_\alpha V^\alpha)$. Therefore, a BRST-invariant generalization of (3.11) is

$$\mathcal{O}_\epsilon(\lambda,\overline{\lambda}) = \int d^{11}f d^{11}\overline{f} d^{11}g d^{11}\overline{g}\ e^{-(\overline{f}_\alpha f^\alpha + \overline{g}_\alpha g^\alpha)}e^{i\epsilon Q(g^\alpha W_\alpha + \overline{f}_\alpha V^\alpha)}\mathcal{O}(\lambda,\overline{\lambda}), \tag{3.22}$$

or, in a more concise way:

$$\mathcal{O}_\epsilon(\lambda,\overline{\lambda}) = \int_{\mathcal{P}} e^{-\overline{f}_\alpha f^\alpha + d\overline{f}_\alpha \wedge df^\alpha} e^{i\epsilon Q(df^\alpha W_\alpha + \overline{f}_\alpha V^\alpha)}\mathcal{O}(\lambda,\overline{\lambda}). \tag{3.23}$$

The integration measure in (3.23) is defined by simply expanding the exponential until the top degree form, i.e. the 22-form, is produced. So in the BRST-invariant version of $\mathcal{O}_\epsilon(\lambda,\overline{\lambda})$, the integration measure is simply $\Omega\overline{\Omega}\Sigma\overline{\Sigma}$ where

$$\Omega = \frac{d^{11}f}{f^3}, \quad \overline{\Omega} = \frac{d^{11}\overline{f}}{\overline{f}^3}, \quad \Sigma = f^3 d^{11}g, \quad \overline{\Sigma} = \overline{f}^3 d^{11}\overline{g}, \tag{3.24}$$

are the top degree forms.

As before, one can show that $\mathcal{O}_\epsilon(\lambda,\overline{\lambda})$ is well-defined at $\lambda = \overline{\lambda} = 0$ as long as $\mathcal{O}(\lambda,\overline{\lambda})$ diverges slower than $(\lambda\overline{\lambda})^{-11}$. And since $\mathcal{O}_\epsilon = \mathcal{O} + \{Q,\chi_\epsilon\}$ for some χ_ϵ, BRST-invariant amplitudes involving $\mathcal{O}_\epsilon$ will be independent of the parameter ϵ.

3.4 Regularized b ghost

The regularization method of (3.22) is easily generalized to the worldsheet operator $b(z)$ of (2.26) by defining

$$b_\epsilon(y) = \int d^{11}f d^{11}\overline{f} d^{11}g d^{11}\overline{g}\ e^{-(\overline{f}_\alpha f^\alpha + \overline{g}_\alpha g^\alpha)}\ b'(y) \tag{3.25}$$

where

$$b'(y) = e^{i\epsilon\{Q,\ g\oint dz U(z) + \overline{f}\oint dz V(z)\}}\ b(y)\ e^{-i\epsilon\{Q,\ g\oint dz U(z) + \overline{f}\oint dz V(z)\}}, \tag{3.26}$$

$U_\alpha(z)$ and $V^\alpha(z)$ are the holomorphic currents defined in (3.20), and the contour integrals in (3.26) go around the point y.

Since $b_\epsilon(y) = b(y) + \{Q,\chi_\epsilon(y)\}$ for some $\chi_\epsilon(y)$, $\{Q,b_\epsilon(y)\} = T(y)$ and BRST-invariant scattering amplitudes are independent of the value of ϵ. Furthermore, since $b(y)$ diverges slower than $(\lambda\overline{\lambda})^{-11}$, $b_\epsilon(y)$ has no singularities at $\lambda(y) = \overline{\lambda}(y) = 0$.

4. Multiloop amplitude prescription

Substituting the regularized b_ϵ ghost of (3.25) for the b ghost, the N-point g-loop amplitude prescription of [4] becomes

$$A = \lim_{\epsilon \to 0} \int d^{3g-3}\tau \left\langle \prod_{j=1}^{3g-3} (\int dw_j \mu_j(w_j) b_\epsilon(w_j)) \prod_{r=1}^{N} \int dz_r U_r(z_r) \, \mathcal{N} \right\rangle \qquad (4.1)$$

where $\mathcal{N}$ is the same regulator for the zero modes as defined in (2.29). For non-zero ϵ, the functional integral is well-defined and, since $b_\epsilon = b + \{Q, \chi_\epsilon\}$ for some χ_ϵ, the amplitude prescription is independent of ϵ up to possible surface terms. So one is free to take the limit $\epsilon \to 0$ after performing the functional integral.

In the multiloop amplitude prescription of (4.1), the functional integral is vanishing unless the integrand contributes 16 θ zero modes for open superstrings, or 32 θ zero modes for closed Type II superstrings. Since the b_ϵ ghost is manifestly spacetime supersymmetric, these θ zero modes can only come either from superfields in the external vertex operators U_r or from the $e^{-(\lambda\bar\lambda + r\theta)}$ term in the zero mode regulator $\mathcal{N}$ of (2.29).

To evaluate (4.1), it is useful to separate the correlation function into two types of terms: terms in which at least one θ zero mode comes from the zero mode regulator $\mathcal{N}$, and terms in which none of the θ zero modes come from the zero mode regulator. As will now be explained, the first type of terms can contribute to F-terms in the ten-dimensional effective action and are easier to evaluate since they do not require ϵ-regularization. The second type of terms are more complicated to evaluate, however, it will be shown that they only contribute near the region $\lambda = \bar\lambda = 0$.

4.1 Ten-dimensional F-terms

Although one does not know how to construct off-shell D=10 superspace actions, one can construct higher-derivative D=10 superspace actions which are functions of on-shell linearized superfields. Ten-dimensional F-terms are defined as manifestly gauge-invariant terms in the superspace effective action which cannot be written as integrals over the maximum number of θ's. In the massless vertex operator for open superstrings, the gauge-invariant superfield of lowest dimension is $W^\alpha(x, \theta)$ whose lowest component is the gluino of dimension $\frac{1}{2}$. Since N=1 D=10 superspace contains 16 θ's, any term in the superspace action involving M superfields W^α which is integrated over the full superspace has dimension $\geq (M + 16)/2$. Therefore, any term in the N=1 D=10 superspace action involving M field-strengths which has dimension less than $(M + 16)/2$ is necessarily an N=1 D=10 F-term.

In the massless vertex operator for closed Type II superstrings, the gauge-invariant superfield of lowest dimension is $W^{\alpha\beta}(x, \theta, \bar\theta)$ whose lowest component is the Ramond-Ramond field strength of dimension 1. Note that the dilaton and axion are dimension zero fields, but they always appear with derivatives in the massless vertex operator. Since N=2 D=10 superspace contains 32 θ's, any term in the superspace action involving M superfields $W^{\alpha\beta}$ which is integrated over the full superspace has dimension $\geq (M + 16)$.

Therefore, any term in the N=2 D=10 superspace action involving M field-strengths which has dimension less than $(M + 16)$ is necessarily an N=2 D=10 F-term. For example, since the curvature tensor R_{mnpq} has dimension 2, the term

$$\int d^{10}x \sqrt{g} \partial^L R^M \tag{4.2}$$

in the Type II effective action is an N=2 D=10 F-term if $L + 2M < M + 16$, i.e. if $L + M < 16$.

If all θ zero modes come from superfields in the external vertex operators in (4.1), the resulting term in the superspace effective action is expressed as an integral over the maximum number of θ's and therefore does not contribute to F-terms. However, if any of the θ zero modes come from $\mathcal{N}$, the resulting term in the superspace effective action is expressed as an integral over a subset of the θ's. Although this does not automatically imply that it is an F-term (since it may be possible to rewrite the expression as an integral over all the θ's), it might contribute to F-terms.

So any term in the scattering amplitude which contributes to an F-term in the effective action must receive at least one θ zero mode from $\mathcal{N}$. It will now be shown that any such term diverges slower than $(\lambda\overline{\lambda})^{-11}$ and therefore does not require ϵ-regularization of the b ghost.

To show that terms receiving θ zero modes from $\mathcal{N}$ do not require ϵ-regularization, first note that BRST invariance implies that the $e^{-(\lambda\overline{\lambda}+r\theta)}$ term in $\mathcal{N}$ can be modified to $e^{-\rho(\lambda\overline{\lambda}+r\theta)}$ for any positive ρ. Because

$$e^{-\rho(\lambda\overline{\lambda}+r\theta)} = e^{\{Q,-\rho\theta\overline{\lambda}\}} = 1 + \{Q, \xi_\rho\} \tag{4.3}$$

for some ξ_ρ, BRST-invariant ampitudes are independent of the value of ρ.

Suppose one computes the amplitude $\langle F(\lambda, \overline{\lambda}) \, \mathcal{N} \rangle$ where $F(\lambda, \overline{\lambda})$ is some BRST-invariant operator. Then ρ-independence implies that the $(-\rho\theta r)^n$ terms in

$$e^{-\rho(\lambda\overline{\lambda}+r\theta)} = e^{-\rho\lambda\overline{\lambda}} \left(1 + \sum_{n=1}^{11} \frac{1}{n!}(-\rho\theta r)^n \right)$$

can only contribute to $\langle F(\lambda, \overline{\lambda}) \, \mathcal{N} \rangle$ if $\int d^{11}\lambda d^{11}\overline{\lambda} \, F(\lambda, \overline{\lambda}) \, e^{-\rho\lambda\overline{\lambda}}$ has poles in ρ. But this implies that $F(\lambda, \overline{\lambda})$ diverges slower than $(\lambda\overline{\lambda})^{-11}$ since

$$\int d^{11}\lambda d^{11}\overline{\lambda} \, (\lambda\overline{\lambda})^{-l} \, e^{-\rho\lambda\overline{\lambda}} \propto \rho^{l-11}. \tag{4.4}$$

So θ zero modes in $\mathcal{N}$ can only contribute to $\langle F(\lambda, \overline{\lambda}) \, \mathcal{N} \rangle$ if $F(\lambda, \overline{\lambda})$ diverges slower than $(\lambda\overline{\lambda})^{-11}$, which implies that ϵ-regularization is unnecessary. So any term which receives θ zero modes from $\mathcal{N}$ can be evaluated by directly setting $\epsilon = 0$ before performing the correlation function.

4.2 Terms requiring ϵ-regularization

For terms in which all θ zero modes come from the vertex operators, ρ-independence of the amplitude implies that (4.4) cannot have poles in ρ, so $l \geq 11$. Therefore, $\int d^{11}\lambda d^{11}\overline{\lambda} \, (\lambda\overline{\lambda})^{-l}$ diverges and ϵ-regularization of the b ghost is necessary. Although the computation of these terms is complicated, integration over the non-minimal fermions r_α will imply that the only contribution to these terms comes from the region near $\lambda = \overline{\lambda} = 0$.

To show that the only contribution come from the region near $\lambda = \overline{\lambda} = 0$, first note that the unregularized b ghost of (2.26) commutes with the conserved charges

$$\oint dz (r_\alpha s^\alpha - \lambda^\alpha w_\alpha) \quad \text{and} \quad \oint dz \overline{\lambda}_\alpha s^\alpha. \tag{4.5}$$

In other words, all terms in the unregularized b ghost have r-charge opposite to their λ-charge, and are invariant under the shift $\delta r_\alpha = c\overline{\lambda}_\alpha$ for constant c. Furthermore, since (4.4) has no poles in ρ, ρ-independence implies that one can directly set $\rho = 0$ in $\mathcal{N}$ so that

$$\mathcal{N}_{\rho=0} = \exp\left(\sum_{I=1}^{g} \left[-\frac{1}{2}N_{mn}^I \overline{N}^{mnI} - J^I \overline{J}^I - \frac{1}{4}S_{mn}^I \, d^I \gamma^{mn}\lambda - S^I \, \lambda^\alpha d_\alpha^I \right] \right). \tag{4.6}$$

One can check that (4.6) also commutes with (4.5), so if the vertex operators U_r are chosen to be independent of the non-minimal variables, the unregularized integrand

$$\prod_{s=1}^{3g-3} b(w_s) \prod_{r=1}^{N} U_r(z_r) \, \mathcal{N}_{\rho=0} \tag{4.7}$$

commutes with the charges of (4.5).

This implies that before performing ϵ-regularization of these terms, the integrand of (4.1) has the form

$$\sum_{k\geq 0} C_k^{\alpha_1 \ldots \alpha_{11+k}} \, \frac{r_{\alpha_1} \ldots r_{\alpha_{11+k}}}{(\overline{\lambda}\lambda)^{11+k}} \tag{4.8}$$

where $C_k^{\alpha_1 \ldots \alpha_{11+k}}$ are operators which carry zero λ-charge and zero r-charge, and which satisfy $\overline{\lambda}_{\alpha_1} C_k^{(\alpha_1 \ldots \alpha_{11+k})} = 0$.

Since r_α has eleven zero modes, at least k of the $(11+k)$ r's in (4.8) must contribute non-zero modes. Furthermore, when $k = 0$, at least one of the eleven r's in (4.8) must contribute a non-zero mode because of the invariance under $\delta r_\alpha = c\overline{\lambda}_\alpha$. So for the correlation function to be non-vanishing, terms coming from the ϵ-regularization must provide non-zero modes of s^α which can contract with these non-zero modes of r_α.

These s_α non-zero modes can come from V^α of (3.20) through the regularization factor $e^{i\epsilon\overline{g}_\alpha \oint dz V^\alpha}$ in b_ϵ, which means that each s^α non-zero mode comes multiplied by a factor of ϵ. So these terms vanish in the limit $\epsilon \to 0$, except near $\lambda = \overline{\lambda} = 0$ where ϵ-regularization can produce poles in ϵ. Therefore, to evaluate terms in which all θ zero modes come from vertex operators, one only needs to evaluate the functional integral $\int d^{11}\lambda d^{11}\overline{\lambda}$ near the point $\lambda = \overline{\lambda} = 0$. It might be possible to explicitly evaluate the contributions of these delta functions at $\lambda = \overline{\lambda} = 0$, however, this will not be attempted here.

Acknowledgments

References

[1] N. Berkovits, *Super-Poincaré covariant quantization of the superstring*, *JHEP* **04** (2000) 018 [`hep-th/0001035`].

[2] N. Berkovits, *Multiloop amplitudes and vanishing theorems using the pure spinor formalism for the superstring*, *JHEP* **09** (2004) 047 [`hep-th/0406055`]; *Covariant multiloop superstring amplitudes*, *Comptes Rendus Physique* **6** (2005) 185–197 [`hep-th/0410079`].

[3] N. Nekrasov, *Lectures at the 23d Jerusalem winter school in theoretical physics*, The Hebrew University Jerusalem, January 2-5, 2006
`http://www.as.huji.ac.il/schools/phys23/media2.shtml`.

[4] N. Berkovits, *Pure spinor formalism as an $N = 2$ topological string*, *JHEP* **10** (2005) 089 [`hep-th/0509120`].

[5] E. Witten, *Two-dimensional models with (0,2) supersymmetry: perturbative aspects*, `hep-th/0504078`.

[6] A.S. Losev, A. Marshakov and A.M. Zeitlin, *On first order formalism in string theory*, *Phys. Lett.* **B 633** (2006) 375 [`hep-th/0510065`].

[7] E. Frenkel and A. Losev, *Mirror symmetry in two steps: A-I-B*, *Commun. Math. Phys.* **269** (2007) 39 [`hep-th/0505131`].

[8] M. Bershadsky, S. Cecotti, H. Ooguri and C. Vafa, *Kodaira-Spencer theory of gravity and exact results for quantum string amplitudes*, *Commun. Math. Phys.* **165** (1994) 311 [`hep-th/9309140`].

[9] I. Antoniadis, E. Gava, K.S. Narain and T.R. Taylor, *Topological amplitudes in string theory*, *Nucl. Phys.* **B 413** (1994) 162 [`hep-th/9307158`].

[10] N. Berkovits and C. Vafa, *$N = 4$ topological strings*, *Nucl. Phys.* **B 433** (1995) 123 [`hep-th/9407190`].

[11] N.A. Nekrasov, *Lectures on curved beta-gamma systems, pure spinors and anomalies*, `hep-th/0511008`.

[12] S. Cordes, G.W. Moore and S. Ramgoolam, *Lectures on 2D Yang-Mills theory, equivariant cohomology and topological field theories*, *Nucl. Phys.* **41** *(Proc. Suppl.)* (1995) 184 [`hep-th/9411210`].

[13] E. Witten, *Two-dimensional gauge theories revisited*, *J. Geom. Phys.* **9** (1992) 303 [hep-th/9204083].

[14] A.S. Schwarz and A.A. Tseytlin, *Dilaton shift under duality and torsion of elliptic complex*, *Nucl. Phys.* **B 399** (1993) 691 [hep-th/9210015].

[15] E.J. Martinec and S.L. Shatashvili, *Black hole physics and Liouville theory*, *Nucl. Phys.* **B 368** (1992) 338.

[16] M. Roček and E.P. Verlinde, *Duality, quotients and currents*, *Nucl. Phys.* **B 373** (1992) 630 [hep-th/9110053].

[17] A.M. Polyakov, *Quantum geometry of bosonic strings*, *Phys. Lett.* **B 103** (1981) 207; *Quantum geometry of fermionic strings*, *Phys. Lett.* **B 103** (1981) 211.

[18] A.A. Belavin, A.M. Polyakov and A.B. Zamolodchikov, *Infinite conformal symmetry in two-dimensional quantum field theory*, *Nucl. Phys.* **B 241** (1984) 333.

[19] N. Seiberg and E. Witten, *String theory and noncommutative geometry*, *JHEP* **09** (1999) 032 [hep-th/9908142].

[20] E. Frenkel, A. Losev, N. Nekrasov, *Instantons and conformal field theories*, to appear.

[21] V.G. Knizhnik, *Analytic fields on riemannian surfaces*, *Phys. Lett.* **B 180** (1986) 247; *Analytic fields on Riemann surfaces II*, *Commun. Math. Phys.* **112** (1987) 567.

[22] D. Friedan, S.H. Shenker and E.J. Martinec, *Covariant quantization of superstrings*, *Phys. Lett.* **B 160** (1985) 55; *Conformal invariance, supersymmetry and string theory*, *Nucl. Phys.* **B 271** (1986) 93.

[23] A. Kotov, P. Schaller and T. Strobl, *Dirac sigma models*, *Commun. Math. Phys.* **260** (2005) 455 [hep-th/0411112].

[24] M. Gualtieri, *Generalized complex geometry*, math.DG/0401221.

[25] N. Hitchin, *Generalized Calabi-Yau manifolds*, math.DG/0209099.

[26] T.J. Courant, *Dirac manifolds*, *Trans. AMS* **319** (1990) 631.

[27] E. Sharpe, *Lectures on D-branes and sheaves*, hep-th/0307245.

[28] P. Griffiths, J. Harris, *Principles of algebraic geometry*, 1978, New York, Wiley & Sons.

[29] V. Gorbounov, F. Malikov, V.Schekhtman, *On chiral differential operators over homogeneous spaces*, math.AG/0008154; *Gerbes of chiral differential operators, I*, math.AG/0005201; *Gerbes of chiral differential operators, II*, math.AG/0003170; *Gerbes of chiral differential operators, III*, math.AG/9906117.

[30] A. Alekseev and T. Strobl, *Current algebra and differential geometry*, *JHEP* **03** (2005) 035 [hep-th/0410183].

[31] F. Malikov, V. Schekhtman, A. Vaintrob, *Chiral de Rham complex*, math.AG/9803041; F. Malikov, V. Schekhtman, *Chiral Poincare duality*, math.AG/9905008; *Chiral de Rham complex II*, math.AG/9901065; V. Gorbounov, F. Malikov, V. Schekhtman, *Deformations of chiral algebras and quantum cohomology of toric varieties*, math.AG/0001170; E. Frenkel, M. Szczesny, *Chiral de Rham complex and orbifolds*, math.ag/0307181; A. Kapustin, *Chiral de rham complex and the half-twisted sigma-model*, hep-th/0504074.

[32] L. Baulieu, A.S. Losev and N.A. Nekrasov, *Target space symmetries in topological theories, I*, *JHEP* **02** (2002) 021 [hep-th/0106042].

[33] J. de Boer, L. Feher, *Wakimoto realizations of current algebras: an explicit construction*, *Commun. Math. Phys.* **189** (1997) 759.

[34] E. Frenkel, *Wakimoto modules, opers and the center at the critical level*, *Adv. in Math.* **195** (2005) 197-404, math.QA/0210029.

[35] B. Fegin, E. Frenkel, *A family of representations of affine Lie algebras*, *Rus. Math. Surveys* **43** (1988) 221; *Representations of affine Kac-Moody algebras and bosonization*, in *Physics and mathematics of strings*, pp. 271-316, World Scientific, 1990; *Representations of affine Kac-Moody algebras, bosonization and resolutions*, *Lett. Math. Phys.* **19** (1990) 307.

[36] B. Feigin, E. Frenkel, *Affine Kac-Moody Algebras and Semi-Infinite Flag Manifolds*, *Commun. Math. Phys.* **128** (1990) 161.

[37] A. Connes, M. Dubois-Violette, *Yang-Mills algebra*, *Lett. Math. Phys.* **61** (2002) 149, math.QA/0206205.

[38] N.A. Nekrasov, *Lectures on open strings and noncommutative gauge fields*, hep-th/0203109.

[39] P.A. Grassi, G. Policastro, M. Porrati and P. Van Nieuwenhuizen, *Covariant quantization of superstrings without pure spinor constraints*, *JHEP* **10** (2002) 054 [hep-th/0112162]; Y. Aisaka and Y. Kazama, *A new first class algebra, homological perturbation and extension of pure spinor formalism for superstring*, *JHEP* **02** (2003) 017 [hep-th/0212316].

[40] K. Liu, *Holomorphic equivariant cohomology*, *Math. Annalen*, **303** (1995) 125.

[41] A. Alekseev, L. Fadeev, S. Shatashvili, *Quantization of symplectic orbits of compact Lie groups by means of the functional integral*, *J. of Geometry and Physics*, **5** (3) (1988) 391.

[42] M. Atiyah, R. Bott, *The moment map and equivariant cohomology* , *Topology* **23** vol. 1(1984) 1.

[43] M. F. Atiyah, I. M. Singer, *Dirac operators coupled to vector bundles*, *Proc. Natl. Acad. Sciences*, **81** (1984) 2597.

[44] L.D. Faddeev and S.L. Shatashvili, *Realization of the schwinger term in the gauss law and the possibility of correct quantization of a theory with anomalies*, *Phys. Lett.* **B 167** (1986) 225; *Algebraic and Hamiltonian methods in the theory of non-abelian anomalies*, *Theor. Math. Phys.* **60** (1984) 770.

[45] N. Nekrasov and S. Shadchin, *ABCD of instantons*, *Commun. Math. Phys.* **252** (2004) 359 [hep-th/0404225].

[46] M. Movshev and A.S. Schwarz, *On maximally supersymmetric Yang-Mills theories*, *Nucl. Phys.* **B 681** (2004) 324 [hep-th/0311132]; *Algebraic structure of Yang-Mills theory*, hep-th/0404183.

[47] A. Gorodentsev and A. Losev, around late 2000, unpublished.

[48] L. Anguelova, P.A. Grassi and P. Vanhove, *Covariant one-loop amplitudes in $D = 11$*, *Nucl. Phys.* **B 702** (2004) 269 [hep-th/0408171]; P.A. Grassi and P. Vanhove, *Topological M-theory from pure spinor formalism*, *Adv. Theor. Math. Phys.* **9** (2005) 285 [hep-th/0411167].

[49] N. Berkovits, *Covariant quantization of the supermembrane*, *JHEP* **09** (2002) 051 [hep-th/0201151].

[50] A.S. Schwarz, *Noncommutative instantons: a new approach*, *Commun. Math. Phys.* **221** (2001) 433 [hep-th/0102182].

[51] E. Witten, *Some properties of O(32) superstrings*, *Phys. Lett.* **B 149** (1984) 351.

[52] M.B. Green and J.H. Schwarz, *Anomaly cancellation in supersymmetric $D = 10$ gauge theory and superstring theory*, *Phys. Lett.* **B 149** (1984) 117.

[53] M.B. Green and J.H. Schwarz, *Covariant description of superstrings*, *Phys. Lett.* **B 136** (1984) 367.

[54] L. Alvarez-Gaumé and E. Witten, *Gravitational anomalies*, *Nucl. Phys.* **B 234** (1984) 269.

[55] G. Moore, P. Nelson, *The aetiology of sigma model anomalies*, *Commun. Math. Phys.* **100** (1985) 83;
P.S. Howe and G. Papadopoulos, *Anomalies in two-dimensional supersymmetric nonlinear sigma models*, *Class. and Quant. Grav.* **4** (1987) 1749.

[56] N. Berkovits, *ICTP lectures on covariant quantization of the superstring*, hep-th/0209059.

[57] N. Berkovits and S.A. Cherkis, *Pure spinors are higher-dimensional twistors*, *JHEP* **12** (2004) 049 [hep-th/0409243].

[58] N. Berkovits, *Multiloop amplitudes and vanishing theorems using the pure spinor formalism for the superstring*, *JHEP* **09** (2004) 047 [hep-th/0406055].

[59] E. Cartan, *The theory of spinors*, Dover, New York, 1981;
P. Budinich and A. Trautman, *The Spinorial Chessboard*, Trieste Notes in Physics, Springer-Verlag, Berlin, 1988;
C. Chevalley, *The algebraic theory of spinors*, Columbia University Press, NY 1954.

[60] N. Berkovits and N. Nekrasov, *The character of pure spinors*, *Lett. Math. Phys.* **74** (2005) 75 [hep-th/0503075].

[61] I. Oda and M. Tonin, *On the b-antighost in the pure spinor quantization of superstrings*, *Phys. Lett.* **B 606** (2005) 218 [hep-th/0409052]; *Y-formalism in pure spinor quantization of superstrings*, *Nucl. Phys.* **B 727** (2005) 176 [hep-th/0505277]; *The b-field in pure spinor quantization of superstrings*, hep-th/0510223.